Asymptotic analysis of direct initiation of gaseous detonations

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In honor to Amable Liñan and Enrique Alarcon
STAMS2019 meeting, Madrid 28-29 March 2019

Gaseous detonations are not easy to initiate

What minimum energy should be deposited
by an external source
to initiate a gaseous detonation in an open space ?

Background in compressible fluids

Planar shock waves and detonations
1860-1940

Detonations = combustion supersonic waves

inert shock followed by an exothermal reaction zone

in gas at ordinary conditions: 1800–3400 m/s, 15–30 bar, 2500–3700 K

Deflagration = combustion subsonic waves

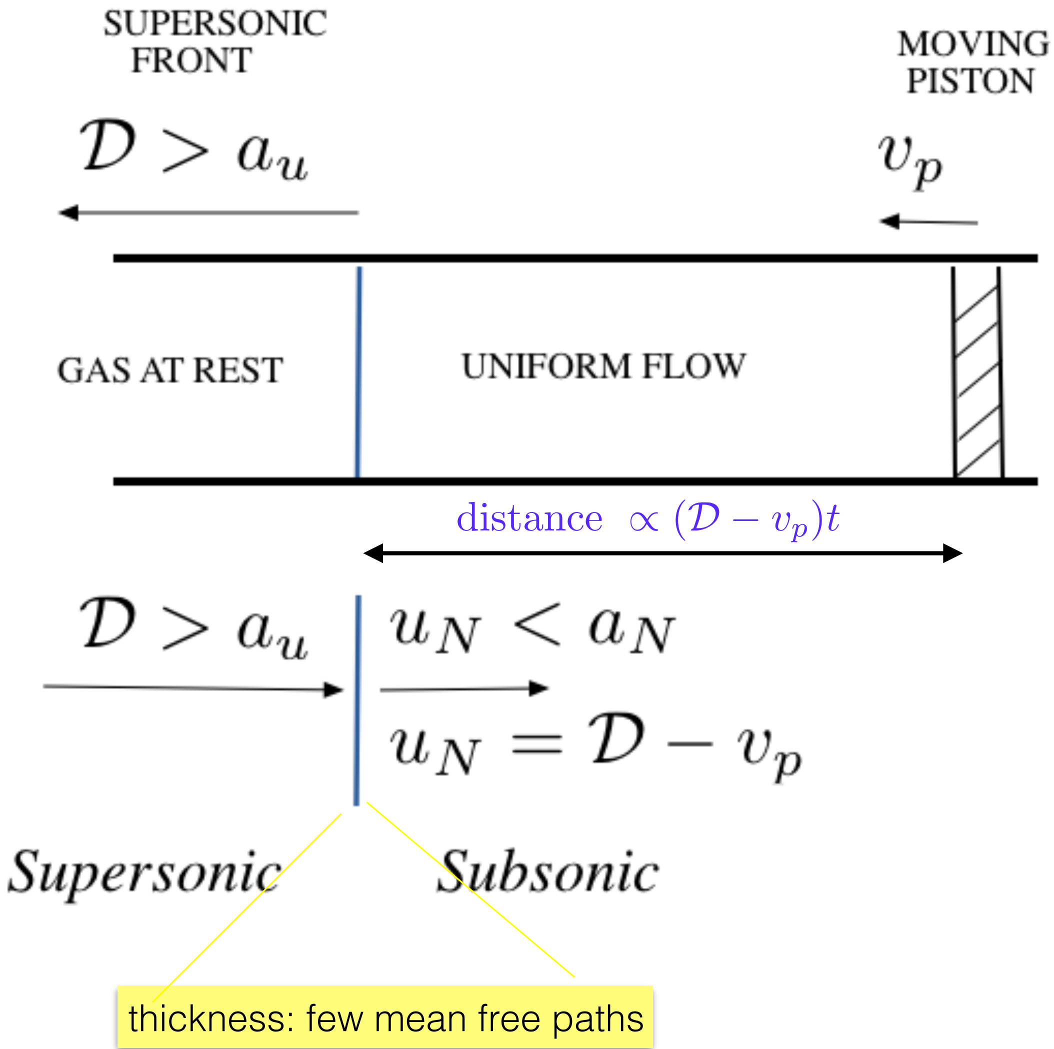
reaction-diffusion waves

Old scientific topic (end of 19th - extensively studied since the mid 20th)

Comprehensible explanation of the multidimensional geometry and complex dynamics of the wave fronts has been elusive. Understanding is recent (nonlinear analyses)

Reactive Euler equations

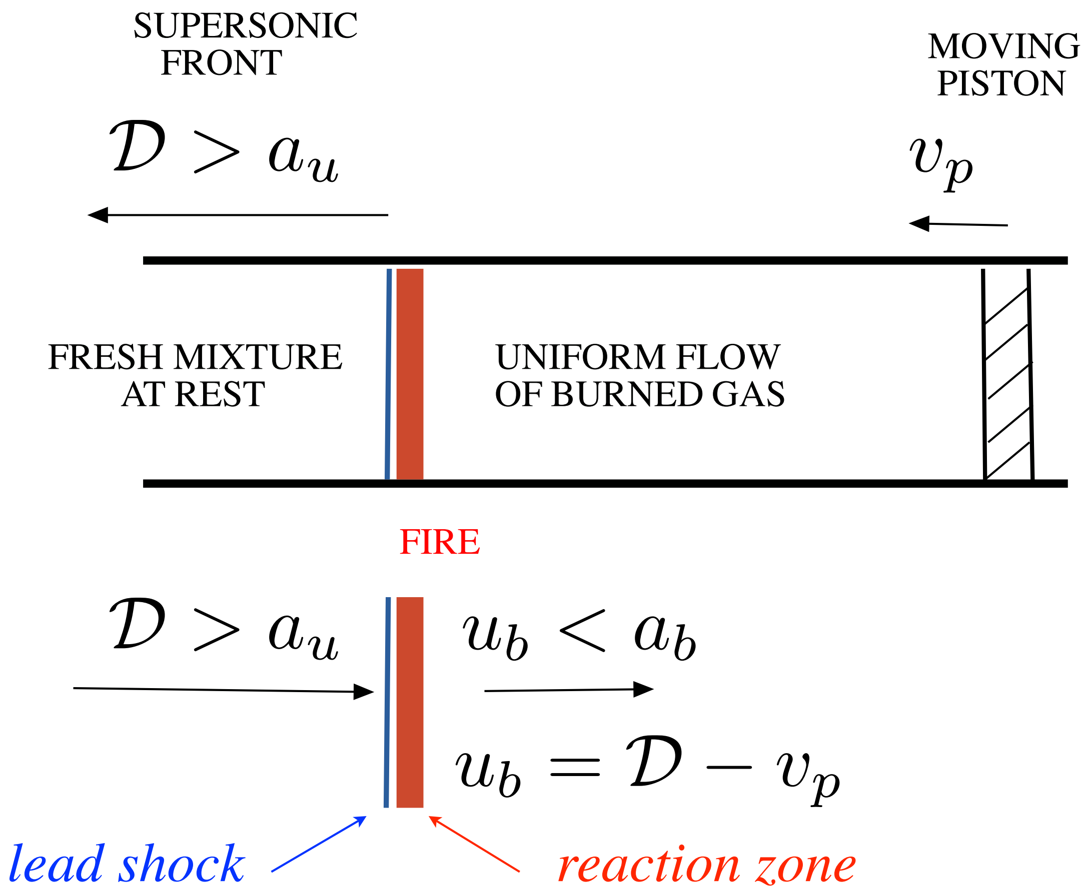
PLANAR SHOCK WAVE
INERT GAS



Poisson 1808, Stokes 1848, **Riemann** 1860, Rankine 1869, **Hugoniot** 1889, Rayleigh 1910

OVERDRIVEN DETONATION
REACTING GAS

PISTON SUPORTED SUPERSONIC WAVE



Abel 1870, Berthelot et Vielle 1881, Mallard et Le Chatelier 1881, **Mikhel'son** 1893, Chapman 1899, Jouguet 1904,

Vielle 1900, **Zel'dovich** 1940, von Neumann1942, Döring 1943,

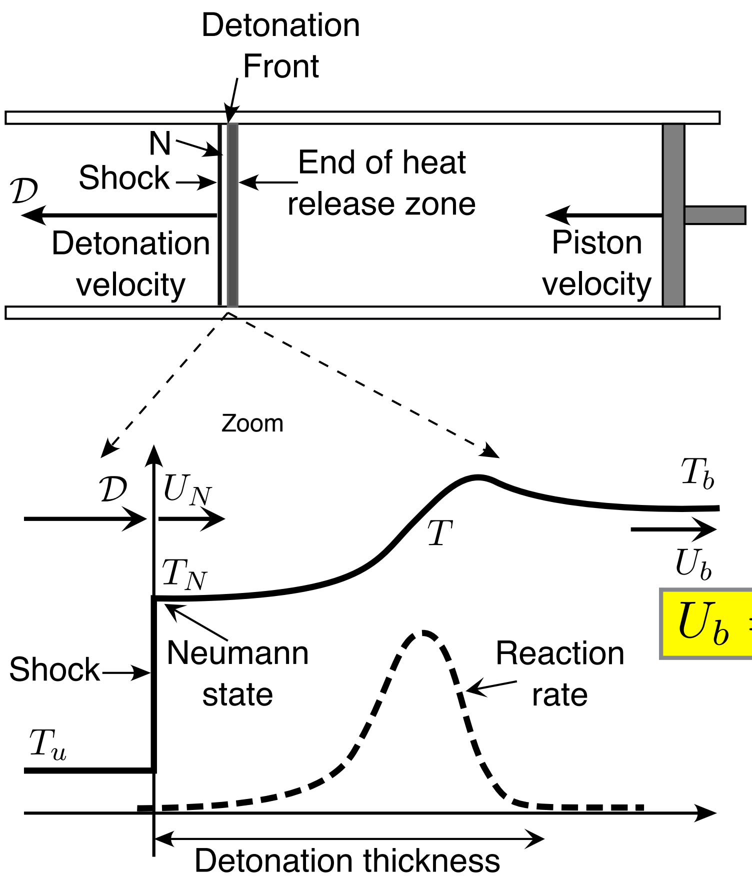
Planar detonation

Overdriven regime / Self sustained wave

Marginal solution

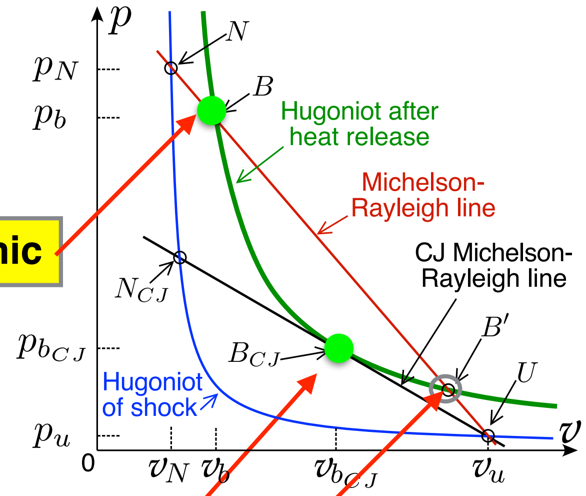
the so called Chapman-Jouguet wave

Mikhel'son (1893)



$U_b = a_b$

subsonic



supersonic without shock
non physical
under ordinary conditions

Sonic condition

Arrhenius law $\quad e^{-E/k_BT} \quad E/k_BT \approx 10$

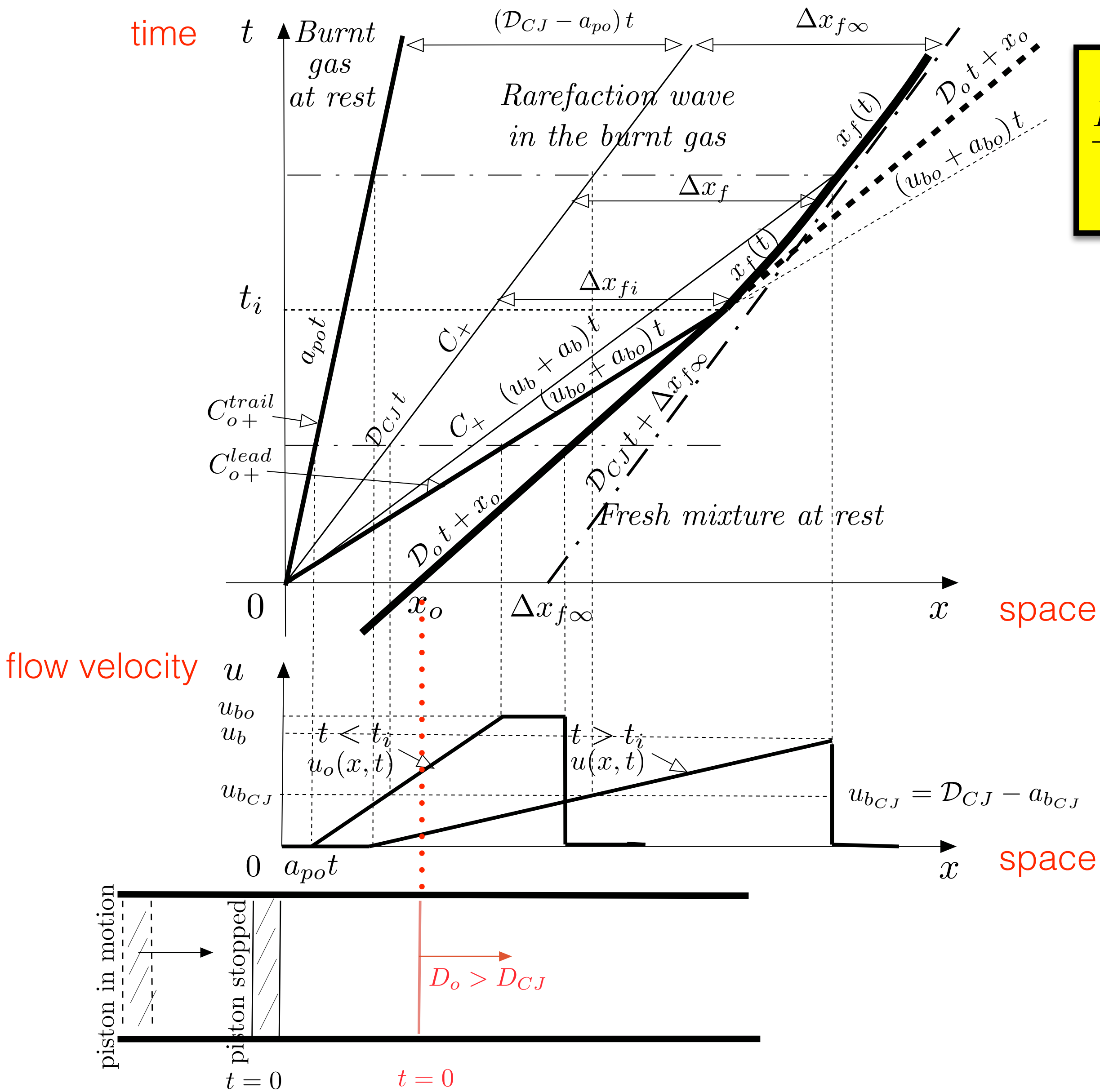
$$reaction\ rate\ 1/t_r \approx e^{-E/k_BT}/t_{coll}$$

$$thickness\ l \approx a_u t_r\ few\ millimeters$$

$$\mathrm{Re} = a_u^2 t_r/\nu \approx t_r/t_{coll} \gg 1 \Rightarrow \text{reactive Euler eqs.}$$

Selection mechanism of the self sustained propagation (CJ wave) by the rarefaction wave in the burnt gas (**planar** geometry and inner structure of the detonation in **steady state**)

Levin & Chernyi (1967), Liñan, Kurdyumov & Sanchez (2012), P.C. Denet (2018)

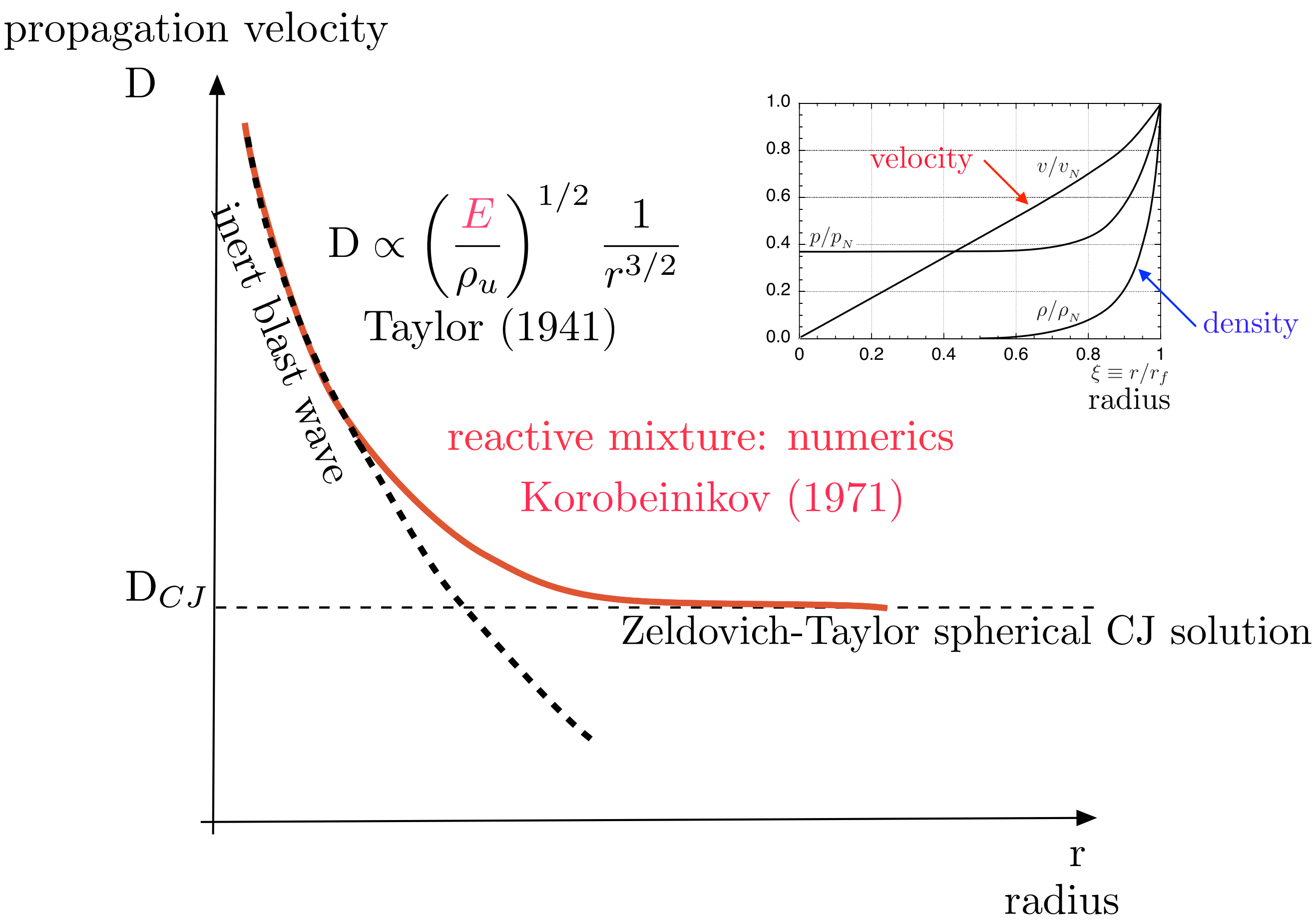


$$\boxed{\frac{D(t) - D_{CJ}}{D_{CJ}} \propto \frac{1}{t^2}}$$

What minimum energy should be released
by an external source
to initiate a gaseous detonation in open space (3D) ?

Numerics in spherical geometry

Korobeinikov (1971)
Detonation = discontinuity
(zero detonation thickness)
No critical energy !



Conclusion: the critical energy is due to modifications of the inner structure

What minimum energy should be released
by an external source
to initiate a gaseous detonation in open space (3D) ?

Zeldovich criterion (1956)

$$\text{critical energy} \;\approx \frac{\text{chemical energy}}{\text{unit volume}} \times (\text{detonation thickness})^3$$

Wrong ! Under estimated by a factor 10^6–10^8

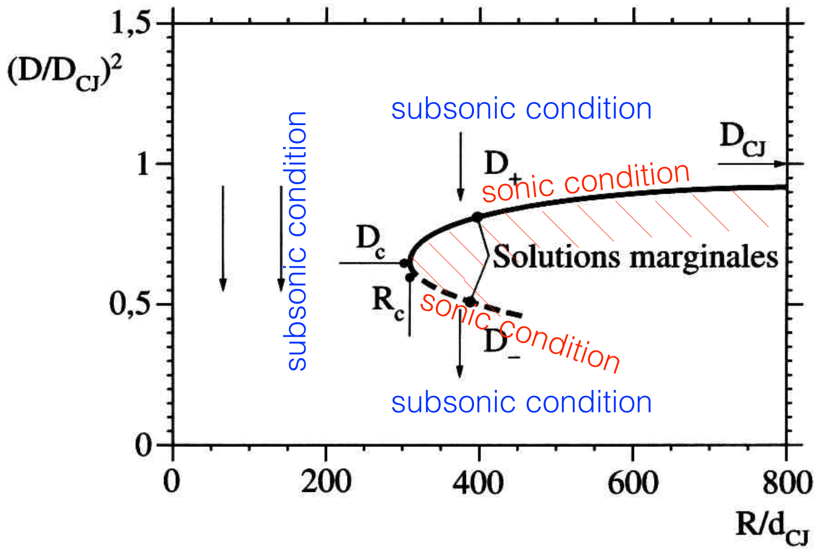
Experiments (1975)-(1980)

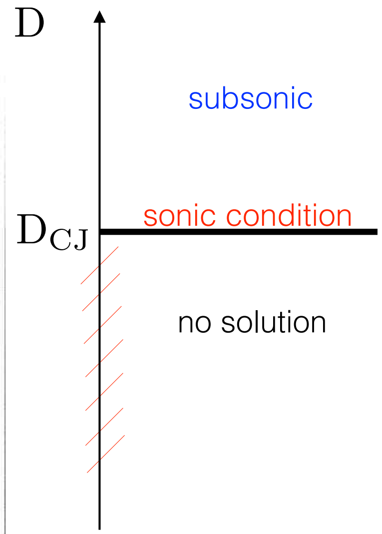
Curvature effect (steady state approximation)

L. He & P.C (1994)

Marginal CJ condition in the parameter space «radius-velocity »

Turning point



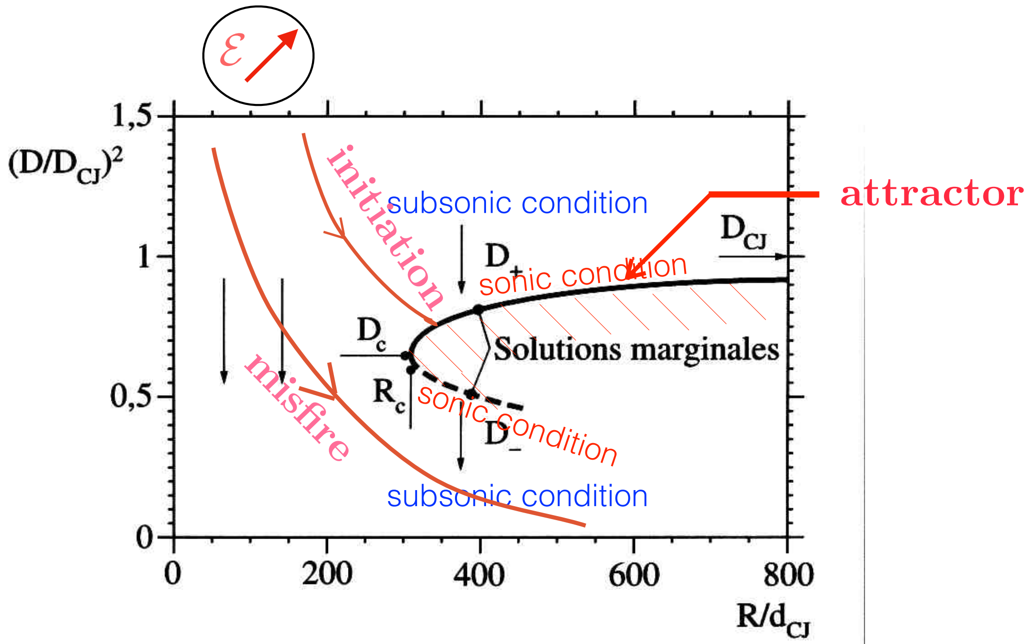


$$\frac{\text{critical radius}}{\text{detonation thickness}} \approx \frac{16\,\mathrm{e}}{(\gamma^2-1)}\frac{E}{k_B T} \approx 10^2$$

Arrhenius factor

Curvature effect (steady state approximation)

Turning point



$$\frac{\text{critical radius}}{\text{detonation thickness}} \approx 10^2 \quad \Rightarrow \quad \boxed{\frac{\text{critical energy}}{\text{Zeldovich value}} \approx 10^6}$$

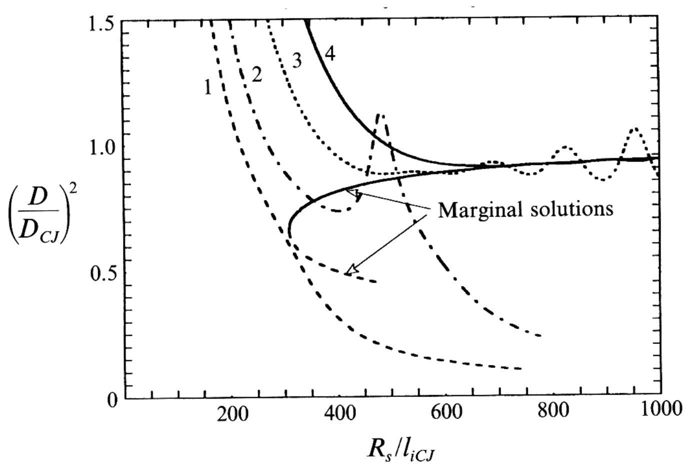
Arrhenius factor

Role of unsteadiness of the inner structure ?
Dynamics close to the critical radius ?

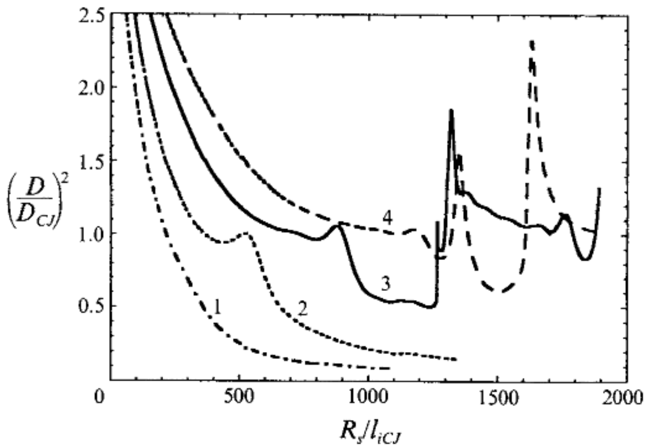
L. He (1995-1996), Lee & Higgins (1999)

DNS show sensitivity to details

Direct initiation of gaseous detonations by an energy source



unsteadiness promotes failure
trajectory 2



unsteadiness promotes re-ignition
trajectory 3

« the primary failure mechanism is found to be unsteadiness »

Eckert, Quirk & Shepherd (2000)

Problem under investigation
not yet fully resolved:

THEORETICAL STUDY OF THE UNSTEADINESS OF THE INNER DETONATION STRUCTURE DURING THE DIRECT INITIATION

GENERAL FORMULATION OF THE PROBLEM

Formulation of the problem of **unsteadiness** in direct initiation
(spherical geometry)

REACTIVE EULER EQUATIONS

$$\frac{1}{\rho}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)\rho+\frac{\partial u}{\partial r}+2\frac{u}{r}=0 \qquad \rho\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)u=-\frac{\partial p}{\partial r} \qquad p=\frac{\gamma-1}{\gamma}c_p\rho T$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)\left[\ln T-\frac{(\gamma-1)}{\gamma}\ln p\right]=\frac{q_m}{c_pT}\frac{\dot{w}(T,Y)}{t_r} \qquad \left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)Y=\frac{\dot{w}(T,Y)}{t_r}$$

CONSERVATIVE FORM: RANKINE-HUGONIOT JUMP CONDITION

$$\frac{u_N}{a_u}=\left(1-\frac{\rho_u}{\rho_N}\right)M$$

$$\frac{p_N}{p_u}=1+\frac{2\gamma}{\gamma+1}(M^2-1) \qquad \frac{\rho_N}{\rho_u}=\frac{1+(M^2-1)}{1+\frac{\gamma-1}{\gamma+1}(M^2-1)}$$

u_N $\longrightarrow$ $\mathcal{D}$ $\longrightarrow$

$M=\mathcal{D}/a_u$

$\rho_N,\, p_N$ | $\rho_u,\, p_u$

shock wave

SELF-SIMILAR SOLUTION: INERT BLAST WAVE $\mathcal{E}$

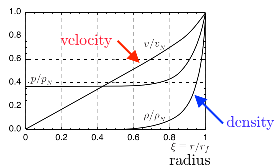
$$\mathcal{D}\equiv\dot{r}_f(t)\propto(\mathcal{E}/\rho_u)^{1/5}t^{-5/3}$$

$$r_f(t)\propto(\mathcal{E}/\rho_u)^{1/5}t^{2/5}$$

lead shock

$$u/\mathcal{D}=U(r/r_f) \qquad \rho/\rho_u=R(r/r_f) \qquad p/(\rho_u\mathcal{D}^2)=P(r/r_f)$$

flow field of the rarefaction wave



Initial condition

Formulation of the problem of **unsteadiness** in direct initiation (spherical geometry)

$$a = \sqrt{\gamma\, p/\rho}$$

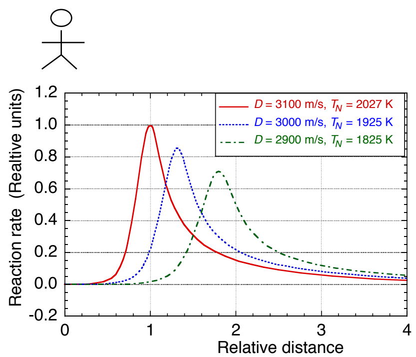
$$\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial r}\right] p \pm \frac{1}{a}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial r}\right] u = \frac{q_m}{c_p T}\frac{\dot{w}}{t_r} - 2\frac{u}{r}$$

acoustic waves

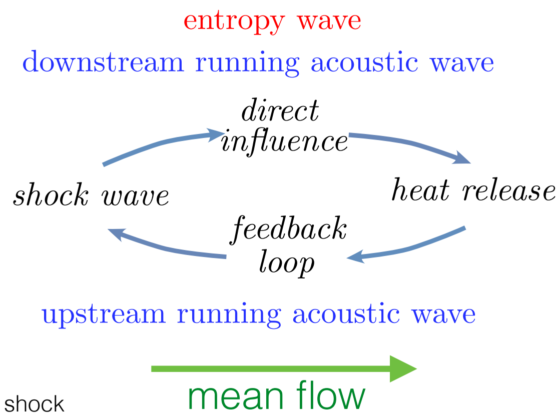
$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\left[\ln T - \frac{(\gamma-1)}{\gamma}\ln p\right] = \frac{q_m}{c_p T}\frac{\dot{w}(T,Y)}{t_r} \qquad \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right) Y = \frac{\dot{w}(T,Y)}{t_r}$$

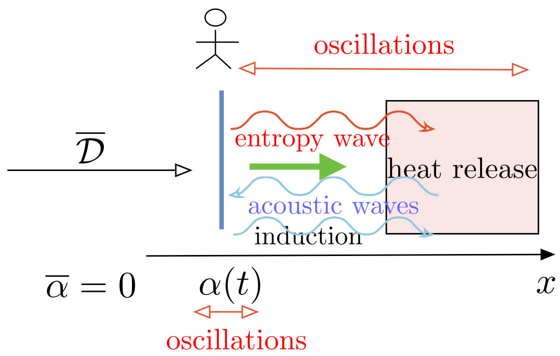
entropy wave

DYNAMICS OF THE INNER STRUCTURE OF THE DETONATION WAVE



distribution of the rate of heat release for different velocity of the leading shock





Formulation of the problem of **unsteadiness** in direct initiation (spherical geometry)

$$\frac{1}{\gamma p}\left[\frac{\partial}{\partial t}+(u\pm a)\frac{\partial}{\partial r}\right]p \pm \frac{1}{a}\left[\frac{\partial}{\partial t}+(u\pm a)\frac{\partial}{\partial r}\right]u = \frac{q_m}{c_p T}\frac{\dot{w}}{t_r} - 2\frac{u}{r}$$

acoustic waves

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)\left[\ln T - \frac{(\gamma-1)}{\gamma}\ln p\right] = \frac{q_m}{c_p T}\frac{\dot{w}(T,Y)}{t_r} \qquad \left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)Y = \frac{\dot{w}(T,Y)}{t_r}$$

entropy wave

Boundary conditions $\quad \xi \equiv \dfrac{r - r_f(t)}{l_o}$

Neumann conditions at the lead shock

$\xi = 0 :\quad$ Rankine-Hugoniot

Condition in the burnt gas

Multiple length-scale pb $\quad l_{ext} \approx r_f \quad l \ll r_f \quad l \ll l_{ext}$

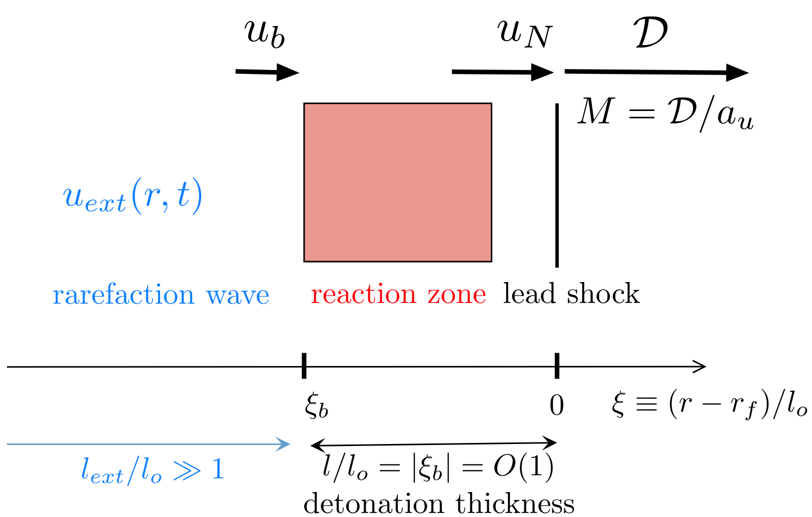
$$\xi = \xi_b(t) : \quad u = u_b(t)$$

exit of the reaction zone

$$u_b(t) \equiv u_{ext}(r_f(t), t)$$

$$\xi < \xi_b : \quad \text{rarefaction wave}$$

$$u = u_{ext}(r,t)$$



SIMPLIFICATIONS

ASYMPTOTIC ANALYSIS

Analytical study of unsteadiness in direct initiation
(spherical geometry)

The problem is too complicated for a general analytical solution

1st simplification:

Taylor approximation

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)\left[\ln T-\cancel{\frac{(\gamma-1)}{\gamma}}\ln p\right]=\frac{q_m}{c_pT}\frac{\dot{w}(T,Y)}{t_r} \qquad \left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)Y=\frac{\dot{w}(T,Y)}{t_r}$$

downstream running $\longleftarrow$

2nd simplification:

$\boxed{a \approx \text{cst.}}$

$$\frac{1}{\gamma p}\left[\frac{\partial}{\partial t}+(u\pm a)\frac{\partial}{\partial r}\right]p\pm\frac{1}{a}\left[\frac{\partial}{\partial t}+(u\pm a)\frac{\partial}{\partial r}\right]u=\frac{q_m}{c_pT}\frac{\dot{w}}{t_r}-2\frac{u}{r}$$

downstream running $\longleftarrow$

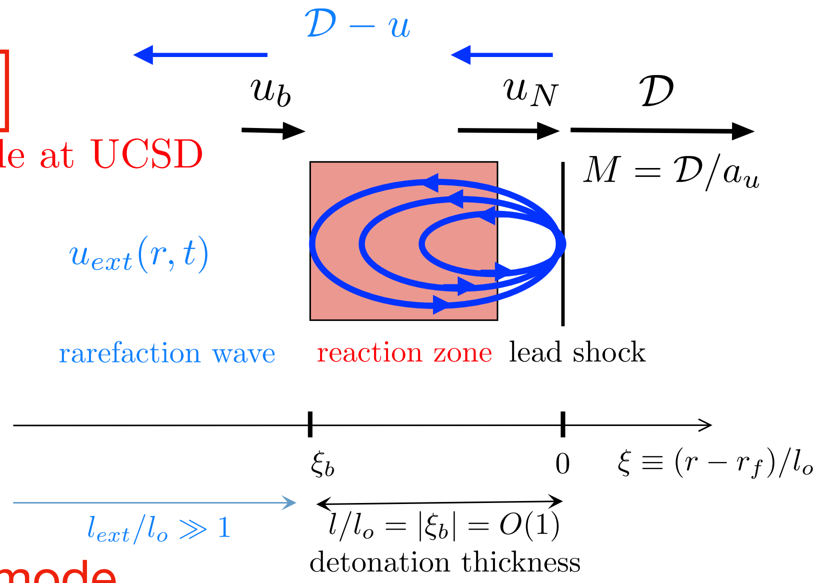
upstream running $\longrightarrow$

3rd simplification:

$\boxed{\text{multiple timescale}}$

following a 1985 suggestion of Amable at UCSD

reference frame attached to the lead shock



downstream running modes
are much faster than
the upstream running mode

The dynamics of the inner structure
is fully controlled by the upstream running acoustic mode

Analytical study of unsteadiness in direct initiation (spherical geometry)

Limit of small heat release

$$\epsilon \equiv (M_{o_{CJ}} - 1) \ll 1 \qquad (\gamma - 1)/\epsilon \ll 1$$

Quasi-transonic flow in the inner structure

flow field

$$\frac{u}{a_u} = \epsilon(1 + \mu(\xi, \tau))$$

shock velocity

$$\frac{\mathcal{D} - \mathcal{D}_{o_{CJ}}}{a_u} \equiv \epsilon\, \dot{\alpha}_\tau(\tau)$$

Unsteady inner structure of the curved detonation

$$\boxed{\begin{aligned}\xi &\equiv (r - r_f)/l_o \\ \tau &\equiv \epsilon\, t/t_r\end{aligned}}$$

curvature

$$\frac{\partial \mu}{\partial \tau} + [\mu - \dot{\alpha}_\tau(\tau)]\frac{\partial \mu}{\partial \xi} = \frac{\dot{w}}{2} - (1 + \mu)\kappa, \qquad l_o/r_f = \epsilon\kappa \approx \text{cst.}$$

$$\boxed{\dot{w} = \mathrm{e}^{b\dot{\alpha}_\tau}\omega_{o_{CJ}}(\xi\, \mathrm{e}^{b\dot{\alpha}_\tau})} \qquad \xi_b(\tau) = -\mathrm{e}^{-b\dot{\alpha}_\tau(\tau)} \qquad \boxed{b = 2(\gamma - 1)\epsilon\frac{E}{k_B T_u}}$$

Boundary conditions

$$\xi = 0 : \mu = 1 + 2\dot{\alpha}_\tau(\tau)$$

RH condition

$$\xi = \xi_b(\tau) : \mu = \mu_b(\tau)$$

prescribed by the rarefaction wave

$$\mu_b(\tau) \geqslant \dot{\alpha}(\tau)$$

CJ regime: $\mu_b(\tau) = \dot{\alpha}(\tau)$

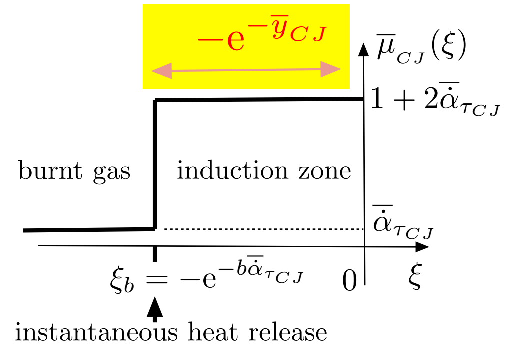
Spherical CJ detonations in steady-state.
CJ peninsula

$$\cancel{\frac{\partial \mu}{\partial \tau}} + [\mu - \dot{\alpha}_\tau(\tau)]\frac{\partial \mu}{\partial \xi} = \frac{\dot{w}}{2} - (1+\overset{\mu_{o_{CJ}}}{\mu})\kappa,$$

$$\xi = 0 : \mu = 1 + 2\dot{\alpha}_\tau(\tau) \qquad \xi = -\mathrm{e}^{-b\dot{\alpha}_\tau(\tau)} : \quad \mu - \dot{\alpha}_\tau = 0$$

sonic condition in the burnt gas

$$(1+\overline{\dot{\alpha}}_{\tau_{CJ}})^2 = 1 - 2\kappa \int_{-\mathrm{e}^{-b\,\overline{\dot{\alpha}}_{\tau_{CJ}}}}^{0} (1+\overline{\mu}_{_{CJ}})\mathrm{d}\xi.$$



Square-wave model: $\overline{\mu}_{_{CJ}}(\xi) = \overline{\mu}_{N_{CJ}} = 1 + 2\overline{\dot{\alpha}}_{_{CJ}}$

$$(1+\overline{\dot{\alpha}}_{\tau_{CJ}})^2 = 1 - 4\kappa(1+\overline{\dot{\alpha}}_{\tau_{CJ}})\mathrm{e}^{-b\,\overline{\dot{\alpha}}_{\tau_{CJ}}}$$

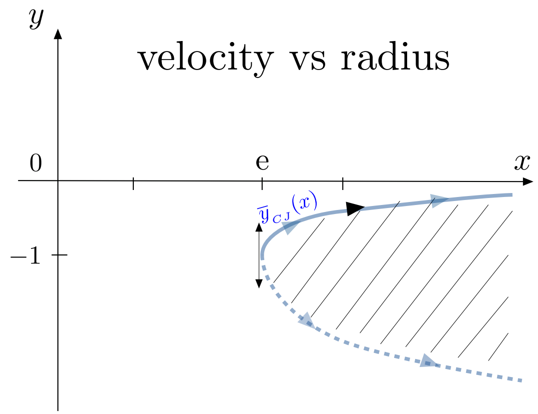
S-shaped curve $\overline{\dot{\alpha}}_{\tau_{CJ}}(\kappa)$ for large b, $b \gtrsim 4$

Asymptotic analysis: $b \gg 1$

$$1/b \ll 1: \quad \kappa = O(1/b), \quad |\overline{\dot{\alpha}}_{\tau_{CJ}}| = O(1/b) \quad \Rightarrow \quad \mathrm{e}^{-b\,\overline{\dot{\alpha}}_{\tau_{CJ}}} = O(1),$$

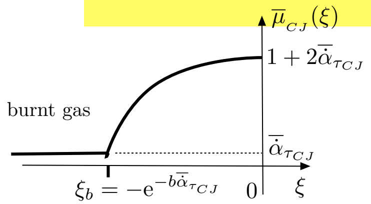
velocity $\nearrow$ $\overline{y}_{_{CJ}} \equiv b\,\overline{\dot{\alpha}}_{\tau_{CJ}} = O(1)$, radius $\nearrow$ $x \equiv \dfrac{1}{2\,b\,\kappa} = O(1)$

C-shaped curve $\overline{y}_{_{CJ}}(x)$: $\overline{y}_{_{CJ}} + \dfrac{1}{x}\mathrm{e}^{-\overline{y}_{CJ}} = 0$



Smooth model:

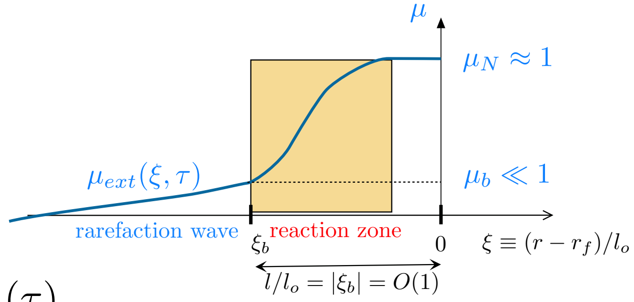
$$x \equiv \frac{1}{\lambda\, b\, \kappa}, \qquad \lambda \equiv 1 + \int_{-1}^{0} \mu_{o_{CJ}}(\xi)\mathrm{d}\xi$$



Analytical study of unsteadiness in direct initiation
(spherical geometry)

Driving mechanism: The rarefaction wave

$$\frac{\partial \mu}{\partial \tau} + [\mu - \dot{\alpha}_\tau(\tau)]\frac{\partial \mu}{\partial \xi} = \frac{\dot{w}}{2} - (1+\mu)\kappa,$$



$\xi = 0 : \mu = 1 + 2\dot{\alpha}_\tau(\tau)$ $\qquad$ $\xi = \xi_b(\tau) : \mu = \mu_b(\tau)$

prescribed by the rarefaction wave

Quasi-transonic flow at the exit of the reaction zone near the **CJ regime** $\quad 0 < \dot{\alpha}_\tau < \mu_{ext} \ll 1$

$$\frac{\partial \mu_{ext}}{\partial \tau} + \cancel{[\mu_{ext} - \dot{\alpha}_\tau(\tau)]\frac{\partial \mu_{ext}}{\partial \xi}} = -\kappa, \qquad \kappa = 1/(b\,\lambda x) = O(1/b)$$

OK with *Liñan et al.* (2012)

$$\boxed{\mathrm{d}\mu_b/\mathrm{d}\tau = -1/(b\,\lambda x)} = O(1/b)$$

local closure condition

slow evolution !

Asymptotic analysis: $b \gg 1$

$$\mathrm{d}r_f/\mathrm{d}t = \mathcal{D} \approx \mathcal{D}_{o_{CJ}} \quad \Rightarrow \quad \mathrm{d}x/\mathrm{d}\tau = 1/(b\,\lambda)$$

$$\boxed{\mu_b = \mu_{bi} - \ln(x/x_i)}$$

Analytical study of unsteadiness in direct initiation (spherical geometry)

Quasi-steady trajectories near the turning point

phase space « velocity-radius »

$$\cancel{\frac{\partial\mu}{\partial\tau}} + [\mu - \dot\alpha_\tau(\tau)]\frac{\partial\mu}{\partial\xi} = \frac{\dot w}{2} - (1+\mu)\kappa,$$

$$\xi = 0 : \mu = 1 + 2\dot\alpha_\tau(\tau) \qquad \xi = -\mathrm{e}^{-b\dot\alpha_\tau(\tau)} : \quad \mu = \mu_b(t)$$

$$\boxed{\kappa = 1/(\lambda x) = O(1/b) \qquad \mu_b = O(1/\sqrt{b})}$$

$$\boxed{\begin{array}{ll} \overline{y} \equiv b\,\overline{\dot\alpha}_\tau = O(1), & x \equiv 1/(2\,b\,\kappa) = O(1) \\ \text{velocity} & \text{radius} \end{array}}$$

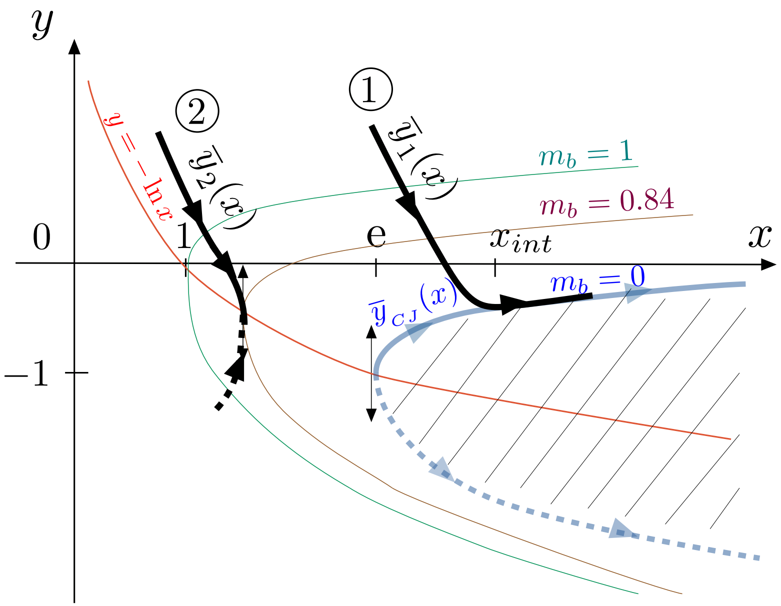
Asymptotic analysis: $b \gg 1$

$$\left(1 + \frac{\overline{y}}{b}\right)^2 = \left(\mu_b - \frac{\overline{y}}{b}\right)^2 + 1 - \frac{2}{b}\frac{e^{-\overline{y}}}{x} \qquad b{\mu_b}^2/2 - \cancel{\overline{y}\mu_b} = \overline{y} + \mathrm{e}^{-\overline{y}}/x$$

$$b{\mu_b}^2/2 = O(1) \quad \Rightarrow \quad \boxed{\mathrm{m}_b \equiv \sqrt{b/2}\,\mu_b = O(1),} \quad \Rightarrow \quad \mathrm{m}_b = \mathrm{m}_{bi} - \sqrt{b/2}\,\ln(x/x_i) \qquad \ln(x/x_i) = O(1/\sqrt{b})$$

$$\boxed{\sqrt{\overline{y}(x) + \mathrm{e}^{-\overline{y}(x)}/x} = \mathrm{m}_{bi} - \sqrt{b/2}\ln(x/x_i)}$$

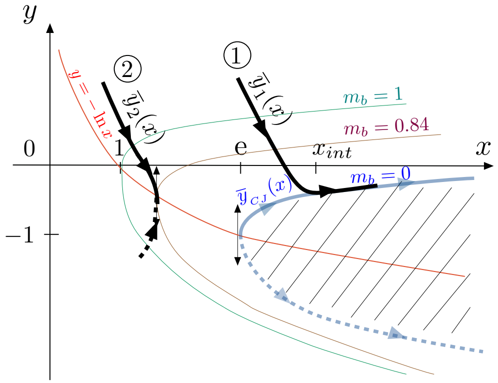
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{1}{(b\,\lambda)} = O(1/b) \qquad \frac{\mathrm{d}\overline{y}}{\mathrm{d}\tau} = O(1/\sqrt{b})$$



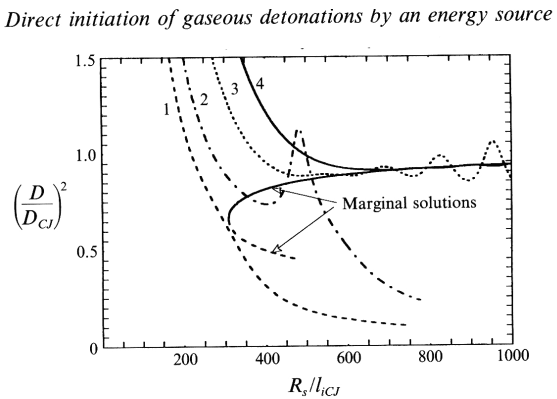
Unfortunately the quasi-steady approximation is not valid

$$\int_{-\mathrm{e}^{-\overline{y}}}^{0} (\partial\mu_0/\partial\tau)\mathrm{d}\xi = -(\lambda - 1)\frac{\mathrm{d}\overline{y}}{\mathrm{d}\tau}e^{-\overline{y}} = O(1/\sqrt{b}),$$

unsteadiness is larger than the curvature effect !

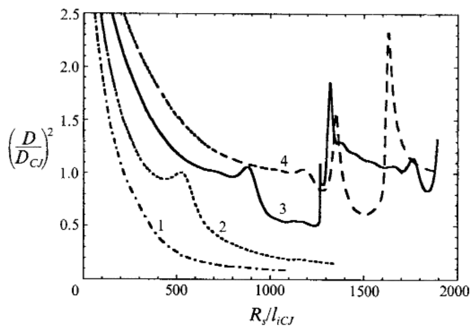


Quasi-steady result for small heat release (2019)



DNS

L. He (1995)-(1996)



Analytical study of unsteadiness for small heat release in the limit of high thermal sensitivity $b \gg 1$

propagation velocity $y(\tau)$ or $y(x)$?

$$\omega(\xi, y) = \mathrm{e}^{y}\omega_{o_{CJ}}(\xi \mathrm{e}^{y})$$

$$\mathrm{d}x/\mathrm{d}\tau = 1/(b\,\lambda)$$

$$\frac{\partial \mu(\xi,\tau)}{\partial \tau} + \left(\mu(\xi,\tau) - \frac{y(\tau)}{b}\right)\frac{\partial \mu(\xi,\tau)}{\partial \xi} = \frac{1}{2}\omega(\xi,\,y) - \frac{1}{b}\,(1+\mu)\,\frac{1}{\lambda\,x(\tau)}$$

$$\xi = 0: \;\; \mu = 1 + 2y(\tau)/b, \qquad \xi = -\mathrm{e}^{-y(\tau)}: \;\; \mu = \sqrt{2}\,\mathrm{m}_b(\tau)/\sqrt{b}$$

$$\mathrm{m}_b = \mathrm{m}_{bi} - \sqrt{b/2}\,\ln(x/x_i)$$

Analytical study of **unsteadiness** in direct initiation
(spherical geometry)

Very difficult problem

The asymptotic analysis is not yet fully completed after 2 years of work
and many fruitful discussions with Amable.

Preliminary results are now presented (not yet published)

Analytical study of unsteadiness for small heat release in the limit of high thermal sensitivity $b \gg 1$

zeroth-order solution

$$\mu(\xi,\tau) = \mu_o(\xi,\tau) + \mu_1(\xi,\tau)$$

$$\frac{\partial \mu_1}{\partial \tau} + \frac{\partial}{\partial \xi}\left[\mu_o \mu_1 + \frac{\mu_1^2}{2}\right] - \cancel{\frac{y(\tau)}{b}}\frac{\partial \mu_1}{\partial \xi} = \frac{1}{b}\mathcal{G}(\xi,\tau) - \cancel{\frac{1}{b}\frac{\mu_1}{\lambda x}}$$

$$J(\xi,\tau) \equiv \frac{\partial \mu_o(\xi,\tau)}{\partial \xi} y(\tau)$$

$$\mathcal{G}(\xi,\tau) \equiv -\frac{1}{\lambda\, x(\tau)} + H(\xi,\tau) + J(\xi,\tau) + K(\xi,\tau) + W(\xi,\tau)$$

$$H(\xi,\tau) \equiv -\frac{\mu_o(\xi,\tau)}{\lambda\, x(\tau)}$$

$$K(\xi,\tau) \equiv -b\,\frac{\partial \mu_o(\xi,\tau)}{\partial \tau} \qquad W(\xi,\tau) \equiv b\,\mathrm{w}(\xi,\tau)$$

$\times\, b\,\mu_o(\xi,\tau)$

$$\mathrm{w}(\xi,\tau) \equiv \frac{1}{2}\left[\mathrm{e}^{y(\tau)}\omega_{o_{CJ}}(\xi\,\mathrm{e}^{y(\tau)})\right] - \underbrace{\mu_o(\xi,\tau)\frac{\partial \mu_o(\xi,\tau)}{\partial \xi}}_{-\frac{1}{2}\left[\mathrm{e}^{y_o(\tau)}\omega_{o_{CJ}}(\xi\,\mathrm{e}^{y_o(\tau)})\right]}$$

$$\xi = 0:\ \ \mu = 1 + 2y(\tau)/b, \qquad \xi = -\mathrm{e}^{-y(\tau)}:\ \ \mu = \sqrt{2}\,\mathrm{m}_b(\tau)/\sqrt{b}$$

$$Z(\xi,\tau) \equiv b[\mu_o\mu_1 + \mu_1^2/2]$$

$\cancel{b\mu_1\partial\mu_1/\partial\tau}$

$$\boxed{\frac{\partial Z}{\partial \tau} + \mu_o\frac{\partial Z}{\partial \xi} - b\mu_1\frac{\partial \mu_o}{\partial \tau} \approx \mu_o(\xi,\tau)\mathcal{G}(\xi,\tau)}$$

Stability analysis of the planar detonation

PC & FA Williams (2002) (2009), *PC & B Denet* (2018)

$$\mu_o \to \overline{\mu}, \qquad \mu_1 \to \delta\mu, \qquad y \to \overline{y} + \delta y$$

$$\mathcal{G}_o \equiv \frac{b}{2}\left[\omega(\xi, y) - \omega(\xi, \overline{y})\right] + \frac{\mathrm{d}\overline{\mu}}{\mathrm{d}\xi}\delta y$$

$$\frac{\partial(\overline{\mu}\,\delta\mu)}{\partial\tau} + \overline{\mu}\frac{\partial(\overline{\mu}\,\delta\mu)}{\partial\xi} = \frac{1}{b}\overline{\mu}(\xi)\,\mathcal{G}_o(\xi, y)$$

$$\xi = 0 : \overline{\mu} = 1 + 2\overline{y}/b, \quad \delta\mu = 2\,\delta y/b, \qquad \xi = -\mathrm{e}^{-\overline{y}} : \quad \delta\mu = 0 \quad (\mathrm{d}\overline{\mu}/\mathrm{d}\xi|_{\xi+\mathrm{e}^{-\overline{y}}=0^-} = 0)$$

$$\overline{\zeta}(\xi) \equiv -\int_{\xi}^{0} \frac{\mathrm{d}\xi'}{\overline{\mu}(\xi')} \qquad \xi(\overline{\zeta}) \text{ is ok} \qquad \overline{\zeta}_b \equiv -\int_{-\mathrm{e}^{-\overline{y}}}^{0} \frac{\mathrm{d}\xi'}{\overline{\mu}(\xi')}$$

$$\frac{\partial(\overline{\mu}\,\delta\mu)}{\partial\tau} + \frac{\partial(\overline{\mu}\,\delta\mu)}{\partial\overline{\zeta}} = \frac{1}{b}\overline{\mu}(\xi)\,\mathcal{G}_o(\xi, y)$$

Integral equation

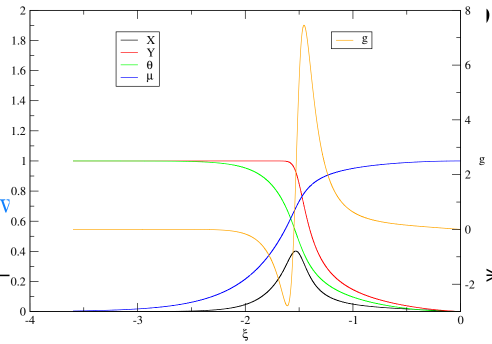
$$2\overline{\mu}(0)\delta y(\tau) = \int_{\overline{\zeta}_b}^{0} \overline{\mu}(\xi)\mathcal{G}_o\big(\xi, \delta y(\tau + \overline{\zeta})\big)\mathrm{d}\overline{\zeta},$$

$$\boxed{2\,\delta y(\tau) = \int_{\overline{\zeta}_b}^{0} \left[\,\overline{\mu}(\xi) g(\xi)\,\right] \delta y\big(\tau + \overline{\zeta}\big)\mathrm{d}\overline{\zeta}}$$

$$g(\xi) \equiv \left\{\frac{b}{2}\frac{\partial}{\partial y}\left[\mathrm{e}^{y}\omega_{o_{CJ}}(e^{y}\xi)\right]_{y=\overline{y}} + \frac{\mathrm{d}\overline{\mu}(\xi)}{\mathrm{d}\xi}\right\}$$

u_b u_N $\mathcal{D}$

$/a_u$



$$\boxed{b_c = 1.27}$$

$- r_f)/l_o$

$l_{ext}/l_o \gg 1$

$l/l_o = |\xi_b| = O(1)$

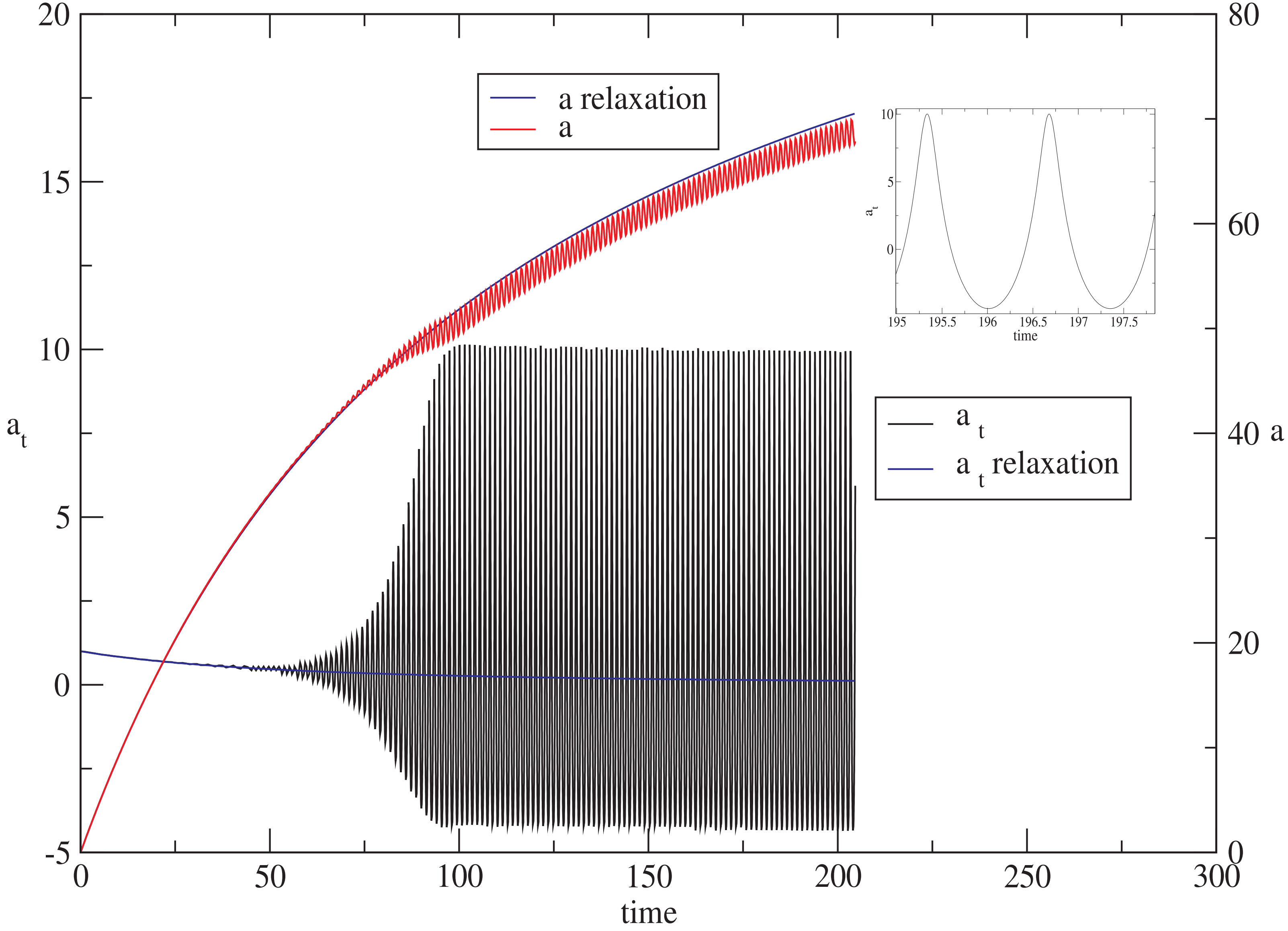
detonation thickness

Oscillatory instability for $b > b_c$ with $b_c = O(1)$

PC & B Denet (2018)

Decay of a planar detonation when the piston is arrested suddenly

PC & B Denet (2018)



Analytical study of unsteadiness for small heat release in the limit of high thermal sensitivity $b \gg 1$

zeroth-order solution

$$\mu(\xi,\tau) = \mu_o(\xi,\tau) + \mu_1(\xi,\tau)$$

$$\frac{\partial \mu_1}{\partial \tau} + \frac{\partial}{\partial \xi}\left[\mu_o\mu_1 + \frac{\mu_1^2}{2}\right] - \cancel{\frac{y(\tau)}{b}}\frac{\partial \mu_1}{\partial \xi} = \frac{1}{b}\mathcal{G}(\xi,\tau) - \cancel{\frac{1}{b}\frac{\mu_1}{\lambda x}}$$

$$J(\xi,\tau) \equiv \frac{\partial \mu_o(\xi,\tau)}{\partial \xi} y(\tau)$$

$$\mathcal{G}(\xi,\tau) \equiv -\frac{1}{\lambda\, x(\tau)} + H(\xi,\tau) + J(\xi,\tau) + K(\xi,\tau) + W(\xi,\tau)$$

MODEL

$$H(\xi,\tau) \equiv -\frac{\mu_o(\xi,\tau)}{\lambda\, x(\tau)}$$

weakly stable and/or unstable detonations

$\beta = O(1)$

$$\boxed{K(\xi,\tau) \equiv -b\,\frac{\partial \mu_o(\xi,\tau)}{\partial \tau} \qquad W(\xi,\tau) \equiv \beta\, \mathrm{w}(\xi,\tau)}$$

moderate unsteadiness

$(\gamma - 1) = O(1/\sqrt{b})$

$$\mathrm{w}(\xi,\tau) \equiv \frac{1}{2}\left[\mathrm{e}^{y(\tau)}\omega_{o_{CJ}}(\xi\,\mathrm{e}^{y(\tau)}) - e^{y_o(\tau)}\omega_{o_{CJ}}(\xi\,\mathrm{e}^{y_o(\tau)})\right]$$

$$\xi = 0:\ \ \mu = 1 + 2y(\tau)/b, \qquad \xi = -\mathrm{e}^{-y(\tau)}:\ \ \mu = \sqrt{2}\,\mathrm{m}_b(\tau)/\sqrt{b}$$

$$Z(\xi,\tau) \equiv b[\mu_o\mu_1 + \mu_1^2/2]$$

$\cancel{\mu_1\partial\mu_1/\partial\tau}$

$$\frac{\partial Z}{\partial \tau} + \mu_o\frac{\partial Z}{\partial \xi} - b\mu_1\frac{\partial \mu_o}{\partial \tau} \approx \mu_o(\xi,\tau)\mathcal{G}(\xi,\tau)$$

$$Z(\xi,\tau) \equiv b[\mu_o\mu_1 + \mu_1^2/2]$$

$$\boxed{\frac{\partial Z}{\partial \tau} + \mu_o\frac{\partial Z}{\partial \xi} - \cancel{b\mu_1\frac{\partial \mu_o}{\partial \tau}} \approx \mu_o(\xi,\tau)\mathcal{G}(\xi,\tau)}$$

time delay

$$\zeta(\xi,\tau) \equiv \int_0^{\xi} \frac{\mathrm{d}\xi'}{\mu_o(\xi',\tau)} < 0, \qquad \frac{\partial \zeta(\xi,\tau)}{\partial \tau} = -\int_0^{\xi} \frac{\partial \mu_o(\xi',\tau)/\partial \tau}{\mu_o^2(\xi',\tau)}\mathrm{d}\xi'$$

$$\mu_o(\xi,\tau)\frac{\partial}{\partial \xi} \to \frac{\partial}{\partial \zeta}, \qquad \frac{\partial}{\partial \tau} \to \frac{\partial}{\partial \tau} + \cancel{\frac{\partial \zeta(\xi,\tau)}{\partial \tau}}\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial \tau} + \frac{\partial \zeta}{\partial \tau}\mu_o(\xi,\tau)\frac{\partial}{\partial \xi}$$

slow dynamics of the zeroth-order solution

$$\boxed{|\partial\zeta(\xi,\tau)/\partial\tau| \ll 1, \qquad \mu_o^{-1}\partial\mu_o(\xi,\tau)/\partial\tau \ll 1,}$$

$$\xi(\zeta,\tau)$$

$$\boxed{\frac{\partial Z(\zeta,\tau)}{\partial \tau} + \frac{\partial Z(\zeta,\tau)}{\partial \zeta} = G(\zeta,\tau)} \quad \text{where} \quad G(\zeta,\tau) \equiv \mu_o(\xi,\tau)\mathcal{G}(\xi,\tau),$$

$$Z(\zeta,\tau) = \int_{\zeta_b}^{\zeta} G(\zeta',\tau+\zeta'-\zeta)\mathrm{d}\zeta' + Z_b(\tau-\zeta+\zeta_b)$$

$$\xi = 0: \ \mu_o + \mu_1 = 1 + 2y(\tau)/b, \qquad \xi = -\mathrm{e}^{-y(\tau)}: \ \mu_o + \mu_1 = \sqrt{2}\,\mathrm{m}_b(\tau)/\sqrt{b}$$

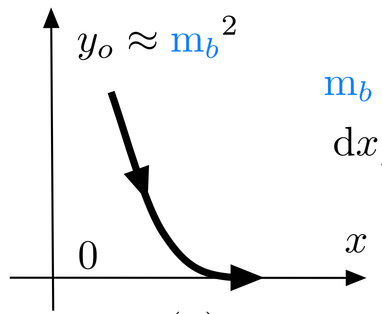
Zeroth-order solution:
Planar overdriven detonation decaying to CJ in the **quasi-steady approximation**

$$\left(\mu_o - \frac{y_o}{b}\right)\frac{\partial \mu_o}{\partial \xi} = \frac{1}{2}\mathrm{e}^{y_o}\omega_{o_{CJ}}(\xi\,\mathrm{e}^{y_o})$$

$$\xi = 0:\ \ \mu_o = 1 + \frac{2y_o}{b}, \qquad \xi = -\mathrm{e}^{-y_o}:\ \ \mu_o = \frac{\sqrt{2}\,\mathrm{m}_b(\tau)}{\sqrt{b}},$$

$$\mu_o(\xi,\tau) - \frac{y_o(\tau)}{b} = \sqrt{\left[\frac{\sqrt{2}\,\mathrm{m}_b(\tau)}{\sqrt{b}} - \frac{y_o(\tau)}{b}\right]^2 + \mu^2_{o_{CJ}}(\xi \mathrm{e}^{y_o(\tau)})}$$

$$y_o(\tau) = \frac{{\mathrm{m}_b}^2(\tau)}{[1+\sqrt{2}\,\mathrm{m}_b(\tau)/\sqrt{b}]} = {\mathrm{m}_b}^2(\tau) + O(1/\sqrt{b})$$



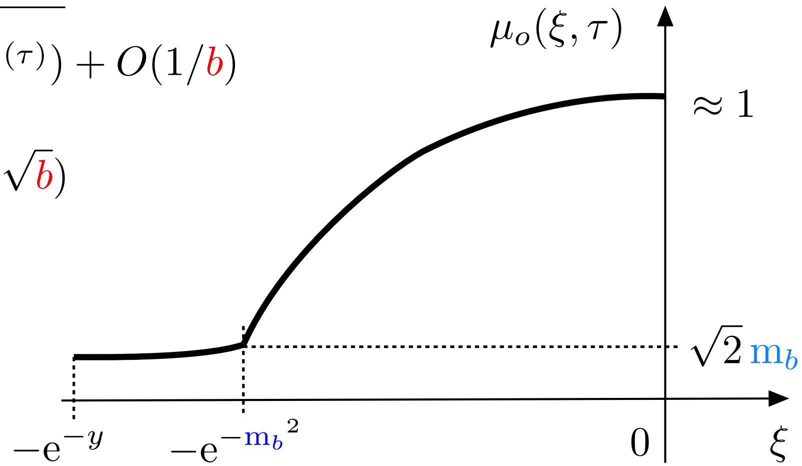
$$\mathrm{m}_b = \mathrm{m}_{bi} - \sqrt{b}\ln(x/x_i)$$

$$\mathrm{d}x/\mathrm{d}\tau = 1/(b\,\lambda)$$

$$\mathrm{m}_b(\tau) \approx \mathrm{m}_{bi} - (\sqrt{2}\lambda)^{-1}(\tau-\tau_i)/\sqrt{b}$$

$$\mu_o(\xi,\tau) = \sqrt{\frac{2{\mathrm{m}_b}^2(\tau)}{b} + \mu^2_{o_{CJ}}(\xi\mathrm{e}^{y_o(\tau)})} + O(1/b)$$

$$\mu_o(\xi,\tau) = \mu_{o_{CJ}}(\xi\mathrm{e}^{{\mathrm{m}_b}^2(\tau)}) + O(1/\sqrt{b})$$

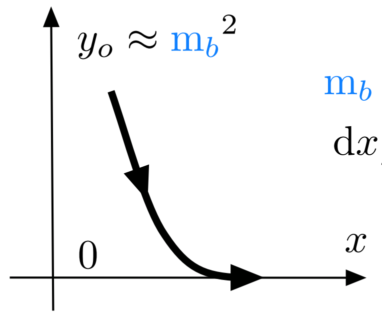


Zeroth-order solution:
Planar overdriven detonation decaying to CJ in the **quasi-steady approximation**

$$\mathrm{m}_b(\tau) \approx \mathrm{m}_{bi} - (\sqrt{2}\lambda)^{-1}(\tau - \tau_i)/\sqrt{b}$$

$$\mu_o(\xi,\tau) - \frac{y_o(\tau)}{b} = \sqrt{\left[\frac{\sqrt{2}\,\mathrm{m}_b(\tau)}{\sqrt{b}} - \frac{y_o(\tau)}{b}\right]^2 + \mu^2_{o_{CJ}}(\xi \mathrm{e}^{y_o(\tau)})}$$

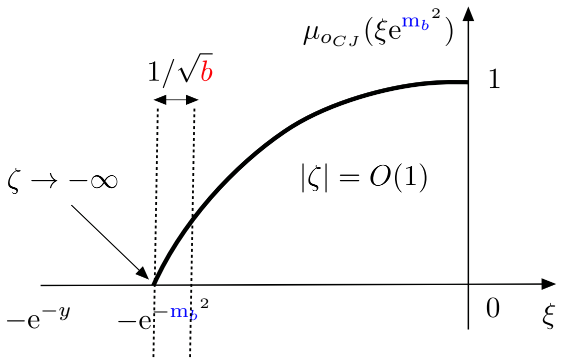
$$y_o(\tau) = \frac{\mathrm{m}_b{}^2(\tau)}{[1 + \sqrt{2}\,\mathrm{m}_b(\tau)/\sqrt{b}]} = \mathrm{m}_b{}^2(\tau) + O(1/\sqrt{b})$$

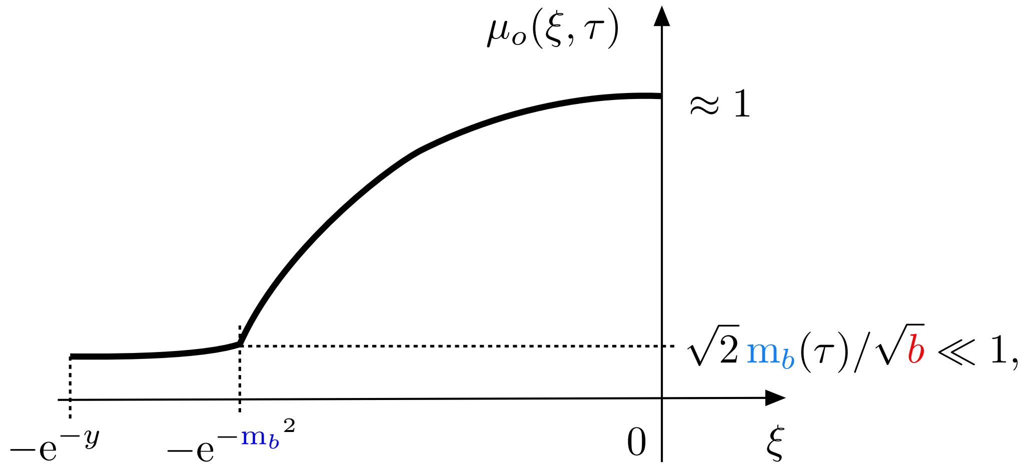


$$\mathrm{m}_b = \mathrm{m}_{bi} - \sqrt{b}\ln(x/x_i)$$
$$\mathrm{d}x/\mathrm{d}\tau = 1/(b\,\lambda)$$

$$\mu_o(\xi,\tau) = \mu_{o_{CJ}}(\xi \mathrm{e}^{\mathrm{m}_b{}^2(\tau)}) + O(1/\sqrt{b})$$

$$\boxed{\mu_o(\xi,\tau) = \sqrt{\frac{2\mathrm{m}_b{}^2(\tau)}{b} + \mu^2_{o_{CJ}}(\xi \mathrm{e}^{y_o(\tau)})} + O(1/b)}$$



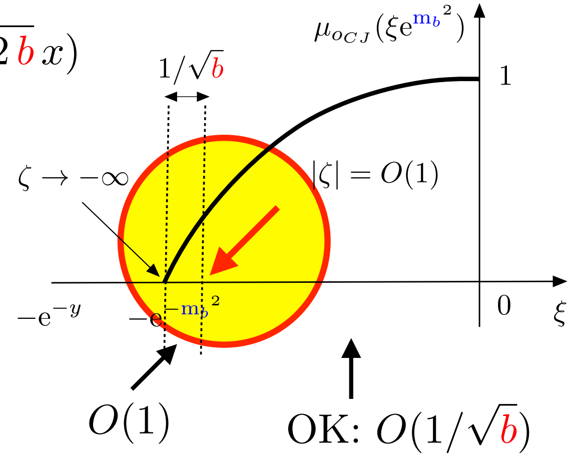


Hot boundary difficulty

(Sonic boundary layer at the end of the reaction in a CJ detonation)

time delay $\zeta(\xi,\tau)\equiv\int_0^{\xi}\dfrac{\mathrm{d}\xi'}{\mu_o(\xi',\tau)}<0,\qquad \dfrac{\partial\zeta(\xi,\tau)}{\partial\tau}=-\int_0^{\xi}\dfrac{\partial\mu_o(\xi',\tau)/\partial\tau}{\mu_o^2(\xi',\tau)}\mathrm{d}\xi'$

$$\mu_o(\xi,\tau)=\mu_{o_{CJ}}(\xi \mathrm{e}^{\mathrm{m}_b{}^2(\tau)})+O(1/\sqrt{b})\qquad \mathrm{d}\mathrm{m}_b/\mathrm{d}\tau=-1/(\sqrt{2}\,b\,x)$$



logarithmic divergence of the time delay

$$\zeta(\xi,\tau)=\int_0^{\xi}\frac{\mathrm{d}\xi'}{\mu_{o_{CJ}}(\xi'\mathrm{e}^{\mathrm{m}_b{}^2(\tau)})}\qquad \lim_{\xi\mathrm{e}^{\mathrm{m}_b{}^2}+1\to0^+}|\zeta|\propto\mathrm{e}^{-\mathrm{m}_b{}^2}\ln(\xi\mathrm{e}^{\mathrm{m}_b{}^2}+1)$$

slow dynamics of the zeroth-order solution ?

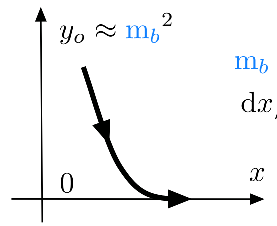
$$|\partial\zeta(\xi,\tau)/\partial\tau|\ll1,\qquad \mu_o^{-1}\partial\mu_o(\xi,\tau)/\partial\tau\ll1,$$

$O(1/\sqrt{b})$ after spatial integration !!

$$Z(\xi,\tau)\equiv b[\mu_o\mu_1+\mu_1^2/2]$$

$$\frac{\partial Z(\zeta,\tau)}{\partial\tau}+\frac{\partial Z(\zeta,\tau)}{\partial\zeta}=G(\zeta,\tau)\quad\text{where}\quad G(\zeta,\tau)\equiv\mu_o(\xi,\tau)\mathcal{G}(\xi,\tau),$$

$$\xi=0:\ \mu_o=1\ \mu_1=2y(\tau)/b,\qquad \xi=-\mathrm{e}^{-y(\tau)}:\ \mu_o=0\ \mu_1=\sqrt{2}\,\mathrm{m}_b(\tau)/\sqrt{b}$$



$\mathrm{m}_b=\mathrm{m}_{bi}-\sqrt{b}\ln(x/x_i)$

$\mathrm{d}x/\mathrm{d}\tau=1/(b\,\lambda)$

$$\zeta=0:\ Z=2y(\tau),\qquad \zeta=-\infty:\ Z=\mathrm{m}_b^2(\tau)$$

$$\mathrm{m}_b(\tau)\approx\mathrm{m}_{bi}-(\sqrt{2}\lambda)^{-1}(\tau-\tau_i)/\sqrt{b}$$

$$Z(\zeta,\tau)=\int_{\zeta_b}^{\zeta}G(\zeta',\tau+\zeta'-\zeta)\mathrm{d}\zeta'+Z(\zeta_b,\tau-\zeta+\zeta_b)\ \Rightarrow\ 2y(\tau)=\int_{\zeta_b}^{0}G(\zeta',\tau+\zeta')\mathrm{d}\zeta'+Z(\zeta_b,\tau+\zeta_b)$$

$|\zeta_b|\to\infty$!!

Skip the difficulty: take the bc outside the thin layer

$$2y(\tau)=\int_{\zeta_b}^{0}G(\zeta',\tau+\zeta')\mathrm{d}\zeta'+\mathrm{m}_b{}^2(\tau)$$

$$\zeta_b+\mathrm{e}^{-\mathrm{m}_b{}^2}=0^+$$

Approximate results using the assumption of a flame structure in quasi-steady state.

This assumption is not justified but the results are encouraging for further theoretical works

$$2y(\tau) = \int_{\zeta_b}^{0} G(\zeta', \tau + \zeta')\mathrm{d}\zeta' + {\mathrm{m}_b}^2(\tau)$$

$$\zeta_b + \mathrm{e}^{-{\mathrm{m}_b}^2} = 0^+$$

$$\int_{\zeta_b}^{0} \mu_o(\xi, \tau)\mathrm{d}\zeta = \int_{-\mathrm{e}^{-{\mathrm{m}_b}^2}}^{0} \mathrm{d}\xi \qquad {\mathrm{m}_b}^2(\tau + \zeta) \approx {\mathrm{m}_b}^2(\tau)$$

$$G(\zeta, \tau) \equiv \mu_o(\xi, \tau)\mathcal{G}(\xi, \tau), \qquad \mu_o(\xi, \tau) = \mu_{o_{CJ}}(\xi \mathrm{e}^{\mathrm{m}_b(\tau)})$$

$$J(\xi, \tau) \equiv \frac{\partial \mu_o(\xi, \tau)}{\partial \xi} y(\tau)$$

$$\mathcal{G}(\xi, \tau) \equiv -\frac{1}{\lambda x} + H(\xi, \tau) + J(\xi, \tau) + K(\xi, \tau) + W(\xi, \tau)$$

$$-\frac{\mathrm{e}^{-{\mathrm{m}_b}^2(\tau)}}{\lambda x}$$

$$H(\xi, \tau) \equiv -\frac{\mu_o(\xi, \tau)}{\lambda x(\tau)}$$

$$I_2 = \int_{-\mathrm{e}^{-{\mathrm{m}_b}^2}}^{0} \mu'_{o_{CJ}}\left(\xi \mathrm{e}^{{\mathrm{m}_b}^2(\tau)}\right) y\big(\tau + \zeta(\xi, \tau)\big)\mathrm{e}^{\mathrm{m}_b(\tau)}\mathrm{d}\xi,$$

$$K(\xi, \tau) \equiv -b\,\frac{\partial \mu_o(\xi, \tau)}{\partial \tau} \qquad W(\xi, \tau) \equiv \beta\, \mathrm{w}(\xi, \tau)$$

$$-\frac{h}{x}\mathrm{m}_b(\tau)\mathrm{e}^{-{\mathrm{m}_b}^2(\tau)}$$

$$\mathrm{w}(\xi, \tau) \equiv \frac{1}{2}\left[\mathrm{e}^{y(\tau)}\omega_{o_{CJ}}(\xi\, \mathrm{e}^{y(\tau)}) - \mathrm{e}^{{\mathrm{m}_b}^2(\tau)}\omega_{o_{CJ}}(\xi\, \mathrm{e}^{{\mathrm{m}_b}^2(\tau)})\right]$$

$$h \equiv \sqrt{2}\, b\, (\lambda - 1)/\lambda = O(1)$$

$$\frac{1}{2}[I_1 - 1] \qquad I_1 = \int_{-\infty}^{0} \mathrm{e}^{y(\tau+\zeta)}\omega_{o_{CJ}}\left(\xi \mathrm{e}^{y(\tau+\zeta)}\right)\mu_{o_{CJ}}\left(\xi \mathrm{e}^{{\mathrm{m}_b}^2(\tau)}\right)\mathrm{d}\zeta$$

Integral equation

$$\mathrm{m}_b(\tau) \approx \mathrm{m}_{bi} - (\sqrt{2}\lambda)^{-1}(\tau - \tau_i)/\sqrt{b}$$

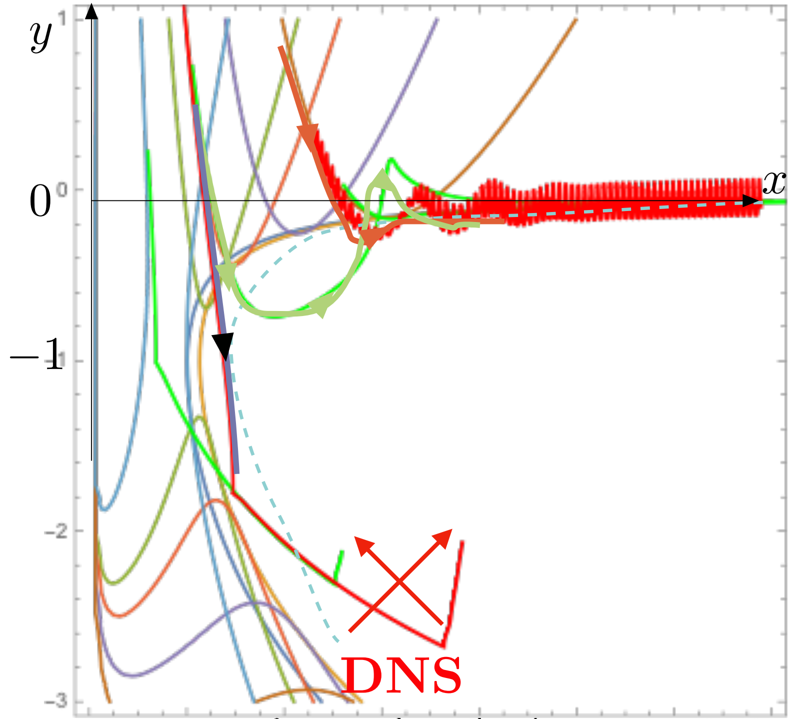
$$y + \frac{1}{x}\mathrm{e}^{-{\mathrm{m}_b}^2(\tau)} = {\mathrm{m}_b}^2(\tau) - \frac{h}{x}\mathrm{e}^{-{\mathrm{m}_b}^2(\tau)}\mathrm{m}_b(\tau) + \underbrace{\frac{\beta}{2}[I_1(y, \tau) - 1] + [I_2(y, \tau) - y]}_{\text{unsteady effects}}$$

$$\underbrace{\frac{1}{x}\mathrm{e}^{-y}}_{\text{quasi-steady solution}}$$

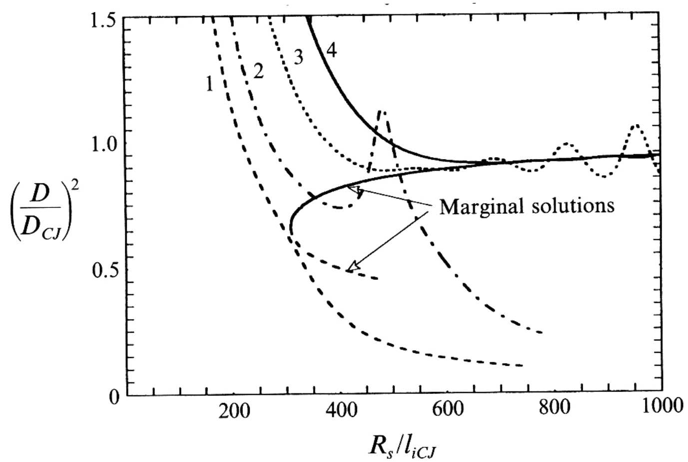
long range memory effect

PRELIMINARY NUMERICAL RESULTS OF THE INTEGRAL EQUATION

Denet 2019



Direct initiation of gaseous detonations by an energy source



CONCLUSION

The preliminary results of the theoretical analysis of unsteadiness in the direct initiation of gaseous detonations are encouraging

The long-range memory effects exhibited in the limit of small heat release are found to be responsible for the main unsteadiness effects

Work is in progress for treating the hot boundary difficulty in a systematic way through an asymptotic limit